SMOOTH MANIFOLDS FALL 2023 - HOMEWORK 8

SOLUTIONS

Problem 1. Find an area form ω on S^2 such that for any rotation about some axis in $\mathbb{R}^3$, $R : S^2 \to S^2$, we have that $R^*\omega = \omega$. Prove the invariance property, and show that this volume form is unique up to positive scalar multiple.

[*Remarks*: You may use the natural coordinates for the tangent spaces as subspaces of $\mathbb{R}^3$ (ie, you may write down a form on $\mathbb{R}^3$ and restrict it to S^2 to construct the form). You may use the fact that rotations around the coordinate axes generate the group of all rotations.]

Solution 1. We claim that the 2-form $\omega = x\,dy \wedge dz - y\,dx \wedge dz + z\,dx \wedge dy$ is invariant under all rotations. The rotations are generated by the vector fields

$$\begin{aligned} X_1 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} \\ X_2 &= z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \\ X_3 &= y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \end{aligned}$$

We will show that $\mathcal{L}_{X_i}\omega = 0$. We verify this for X_1, the equations for X_2 and X_3 follow similarly:

$$\begin{aligned} d\omega &= 3\,dx \wedge dy \wedge dz \\ \iota_{X_1} d\omega &= 3y\,dy \wedge dz + 3x\,dx \wedge dz \\ \iota_{X_1}\omega &= (-y^2\,dz + yz\,dy) - (x^2\,dz - xz\,dx) \\ &= xz\,dx + yz\,dy - (x^2+y^2)\,dz \\ d\iota_{X_1}\omega &= x\,dz \wedge dx + y\,dz \wedge dy - 2y\,dy \wedge dz - 2x\,dx \wedge dz \\ &= -3x\,dx \wedge dz - 3y\,dy \wedge dz \\ \mathcal{L}_{X_1}\omega &= \iota_{X_1} d\omega + d\iota_{X_1}\omega \\ &= 0. \end{aligned}$$

The calcluation for $\mathcal{L}_{X_2}$ and $\mathcal{L}_{X_3}$ is analogous. To see that ω is unique up to scalar multiple, fix any point $x_0 \in S^2$ and let β be any other 2-form invariant under all rotations R. Then there exists a unique $c > 0$ such that $\beta(x_0) = c\omega(x_0)$, since $\dim \Lambda^2(T_{x_0}) = \binom{2}{2} = 1$. If $x \in S^2$ is any other point, there exists a rotation R such that $R(x) = x_0$. Then

$$\begin{aligned} \beta(x)(v_1, v_2) = R^*\beta(x)(v_1, v_2) &= \beta(R(x))(dR(v_1), dR(v_2)) = \beta(x_0)(dR(v_1, dR(v_2)) \\ &= c\omega(x_0)(dR(v_1), dR(v_2)) = (R^{-1})^*\omega(x_0)(dR(v_1), dR(v_2)) = c\omega(x)(v_1, v_2) \end{aligned}$$

Hence $\beta = c\omega$ as 2-forms. ☐

Solution 2. With ω as in Solution 1, we verify $R^*\omega = \omega$ directly for rotations about the z-axis

$$R = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Written as a function,

$$R(x, y, z) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z).$$

Then

$$R^*\omega(x, y, z)(v_1, v_2) = \omega(x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z)(Rv_1, Rv_2).$$

Where Rv_1 denotes R acting as a matrix. It follows that

$$\begin{aligned} R^*\omega = &(x\cos\theta - y\sin\theta)\, d(x\sin\theta + y\cos\theta) \wedge dz \\ &- (x\sin\theta + y\cos\theta)\, d(x\cos\theta - y\sin\theta) \wedge dz + z\, d(x\cos\theta - y\sin\theta) \wedge d(x\sin\theta + y\cos\theta) \end{aligned}$$

Gathering like terms, we see that the $dx \wedge dz$ term is coefficient to

$$(x\cos\theta - y\sin\theta) \cdot \sin\theta - (x\sin\theta + y\cos\theta) \cdot \cos\theta = -y(\sin^2\theta + \cos^2\theta) = -y.$$

The $dy \wedge dz$ coefficent is

$$(x\cos\theta - y\sin\theta) \cdot \cos\theta + (x\sin\theta + y\cos\theta) \cdot \sin\theta = x(\cos^2\theta + \sin^2\theta) = x.$$

The $dx \wedge dy$ term is

$$z[\cos\theta \cdot \cos\theta - (-\sin\theta \cdot \sin\theta)] = z(\cos^2\theta + \sin^2\theta) = z$$

It follows that $R^*\omega = \omega$. The proof of uniqueness follows as in Solution 1. $\square$

Problem 2.

(1) Prove the following convenient formula for the exterior derivative of a 1-form α, where X and Y are vector fields on M. Note that if $f \in C^\infty(M)$, then $X \cdot f$ denotes the derivative of f along the vector field X.

$$d\alpha(X, Y) = X \cdot (\alpha(Y)) - Y \cdot (\alpha(X)) - \alpha([X, Y])$$

[*Hint*: Any 1-form is locally a linear combination of forms of the form $u\, dv$ for $u, v \in C^\infty$. Evaluate both side of the desired equality for such forms.]

(2) Use this to find a formula for $\mathcal{L}_X\alpha(Y)$, where X and Y are C^∞ vector fields and α is a 1-form on M. Think magically!

Solution. Consider a form $\alpha = u\, dv$. Then $d\alpha = du \wedge dv$, so

$$d\alpha(X, Y) = du(X)dv(Y) - du(Y)dv(X) = (X \cdot u)(Y \cdot v) - (Y \cdot u)(X \cdot v)$$

On the other hand,

$$\begin{aligned} X\cdot(\alpha(Y)) &= X\cdot(u\cdot(Y\cdot v)) \\ &= (X\cdot u)\cdot(Y\cdot v)+u\cdot(XYv) \\ -Y\cdot(\alpha(X)) &= -Y\cdot(u\cdot(X\cdot v)) \\ &= -(Yu)\cdot(Xv)-u\cdot(YXv) \\ -\alpha([X,Y]) &= -u(XYv-YXv) \end{aligned}$$

Adding each of these terms shows the desired equality.
Finally,

$$\begin{aligned} \mathcal{L}_X\alpha(Y) &= \iota_X d\alpha(Y)+d(\iota_X\alpha)(Y) \\ &= X\cdot(\alpha(Y))-Y\cdot(\alpha(X))-\alpha([X,Y])+Y\cdot(\alpha(X)) \\ &= X\cdot(\alpha(Y))-\alpha([X,Y]). \end{aligned}$$

□

Problem 3. Fix a $2n$-dimensional manifold M. Recall that a 2-form ω is called *symplectic* if $d\omega = 0$ and the n-fold wedge product of ω is a volume form on M.

(1) Show that a closed 2-form ω is symplectic if and only if for every $x \in M$ and nonzero vector $X \in T_xM$, there exists $Y \in T_xM$ such that $\omega(X,Y) \neq 0$.
(2) Show that if α is any 1-form on M, there exists a unique vector field X_α such that $\iota_{X_\alpha}\omega = \alpha$.

Solution.

(1) First assume that ω is symplectic, so that $\omega^{\wedge n} := \omega\wedge\omega\cdots\wedge\omega$ is nonzero. Then for every nonzero vector field $X \in T_XM$, $\iota_X\omega^{\wedge n} \neq 0$, since every vector can be extended to a basis, which evaluates to a nonzero number on every top form. By induction, we claim that

$$\iota_X\omega^{\wedge n} = n(\iota_X\omega)\wedge(\omega^{\wedge n-1}) \tag{†}$$

Indeed, the base case of $n = 1$ follows immediately. Then assuming the formula for $n-1$,

$$\begin{aligned} \iota_X\omega^{\wedge n} = \iota_X(\omega\wedge\omega^{\wedge n-1}) &= \iota_X\omega\wedge\omega^{\wedge n-1}+\omega\wedge(\iota_X\omega^{\wedge n-1}) \\ &= (\iota_X\omega)\wedge\omega^{\wedge n-1}+\omega\wedge((n-1)(\iota_X\omega)\wedge\omega^{\wedge n-2}) = n(\iota_X\omega)\wedge\omega^{\wedge n-1} \end{aligned}$$

By (†), it follows that since $\iota_X\omega^{\wedge n}\neq 0$, $\iota_X\omega \neq 0$. Hence the nondegeneracy condition follows.

Now,we assume that ω satisfies the nondegeneracy condition, and prove that that $\omega^{\wedge n}\neq 0$. We prove this by finding a basis $\{X_1,Y_1,\ldots,X_n,Y_n\}$ of T_xM such that $\omega^{\wedge n}(X_1,Y_1,\ldots,X_n,Y_n) = 1$. We do this by inductively showing that we may find $\{X_1,Y_1,\ldots,X_k,Y_k\}$ such that for every pair of basis elements V,W,

$$\omega(V,W) = 0 \text{ unless } V = X_i, W = Y_i \text{ for some } i = 1,\ldots,k \tag{I1}$$

and

$$\omega(X_i,Y_i) = 1 \text{ for all } i = 1,\ldots,k. \tag{I2}$$

The base case of $k = 1$ is exactly the nondegeneracy condition described, after rescaling Y as $\frac{Y}{\omega(X,Y)}$. So we assume we have chosen $\{X_1, Y_1, \ldots, X_k, Y_k\}$ and seek to find X_{k+1} and Y_{k+1}. Define the following linear map from T_xM to itself:

$$\phi_x^{(k)} : T_xM \to T_xM \qquad \phi_x^{(k)}(v) = v + \left(\sum_{i=1}^{k} \omega(v, X_i)Y_i - \omega(v, Y_i)X_i\right)$$

We claim that $\phi_x^{(k)}$ is a projection, so that $T_xM = \ker \phi_x^{(k)} \oplus \operatorname{im} \phi_x^{(k)}$ and $\phi_x^{(k)}|_{\operatorname{im} \phi_x^{(k)}} = \mathrm{Id}$. Indeed, observe that by the induction hypotheses (I1) and (I2),

$$\phi_x^{(k)}(X_j) = X_j - X_j = 0 \qquad \phi_x^{(k)}(Y_j) = Y_j - Y_j = 0$$

Hence $H_k := \operatorname{span}_{\mathbb{R}} \{X_1, Y_1, \ldots, X_k, Y_k\} \subset \ker \phi_x^{(k)}$. However, if $v \in \ker \phi_x^{(k)}(v)$, then

$$v = \sum_{i=1}^{k} \omega(v, Y_i)X_i - \omega(v, X_i)Y_i$$

so that $v \in H_k$. It follows that $\ker \phi_x^{(k)} = H_k$, and the image must be a $(2n - 2k)$-dimensional subspace transverse to H_k. Next, note that

$$\phi_x^{(k)}(\phi_x^{(v)}) = \phi_x^{(k)}(v + \Sigma) = \phi_x^{(k)}(v)$$

where Σ is the sum in the definition of $\phi_x^{(k)}$ which belongs to its kenrel. Hence $\phi_x^{(k)}$ is the identity on its image.

Finally, if $L_k = \operatorname{im} \phi_x^{(k)}$, we claim that ω is nondegenerate when restricted to L_k. Indeed, fix $X_{k+1} \in L_k$, and assume for contradiction that $\omega(X_{k+1}, v) = 0$ for all $v \in L_k$. Since $T_xM = H_k \oplus L_k$, any $w \in T_xM$ can be written as $w_H + w_L$, where $w_H \in H_k$ and $w_L \in L_k$. Because $X_{k+1} \in L_k$, $\omega(X_{k+1}, X_i) = \omega(X_{k+1}, Y_i) = 0$ (it follows from direct computation that any $X \in L_k = \operatorname{im} \phi_x^{(k)}$ must satisfy this property). Therefore, for any $w \in T_xM$

$$\omega(X_{k+1}, w) = \omega(X_{k+1}, w_L) + \omega(X_{k+1}, w_H) = 0$$

where one vanishing follows from the preceding observation, and the other follows from the contradiction assumption. Since we have contradicted the nondegeneracy of ω on T_xM, we conclude that ω must also be nondegenerate on L_k. This allows us to pick two vectors X_{k+1} and Y_{k+1} satisfying the desired properties.

The result follows.

(2) Observe that the nondegeneracy condition implies that the map $\Psi : \mathfrak{X}(M) \to \Omega^1(M)$ defined by $\Psi(X) = \iota_X\omega$is a linear isomorphism, where $\mathfrak{X}(M)$ denotes the space of vector fields. Indeed, it suffices to show that the map is an isomorphism pointwise. Fixing $x \in M$, notice that the map $\Psi_x : T_xM \to T_x^*M$ defined by $\Psi_x(X)(v) = \omega(X, v)$ is linear and the domain and range have the same dimension. Furthermore, the nondegeneracy condition from the previous problem implies that $\ker \Psi_x = \{0\}$. Hence Ψ_x is an isomorphism for every x, and Ψ is an isomorphism on the space of sections, with inverse

$$\Psi^{-1}(\alpha)(x) = (\Psi_x)^{-1}(\alpha(x)).$$

The vector field X_α is exactly $\Psi^{-1}(\alpha) \in \mathfrak{X}(M)$.

□